9 Rossby Waves

(Holton Chapter 7, Vallis Chapter 5)

9.1 Non-divergent barotropic vorticity equation

We are now at a point that we can discuss our first fundamental application of the equations of motion: *non-divergent barotropic Rossby waves*! For the derivation of these waves, we will use the simplifying assumptions of

- two-dimensional flow
- barotropic flow

Even though these are rather strong approximations, the solutions turn out to be (surprisingly) relevant to the real atmosphere - and provide deep insight into large-scale midlatitude dynamics. Recall that, in essence, these assumptions are tied to the assumptions that the flow is in near hydrostatic and geostrophic balance (recall that geostrophic flow is non-divergent to leading order on pressure surfaces).

As shown in the previous section, under these assumptions the prognostic equation for vorticity reads:

$$\boxed{\frac{D\zeta_a}{Dt} = \frac{D}{Dt}(\zeta + f) = 0} \tag{9.1}$$

That is, the flow is governed by *absolute vorticity conservation*!

As discussed in lecture, this equation was used to provide the first numerical weather forecast! It was a 24-hour forecast (looking forward 24-hours) that took 24-hours to complete! However, it was considered an ultimate success!

9.1.1 Preparing to solve the vorticity equation

Step 1: Linearization

Although 9.1 appears simple, we are not generally able to solve it an analytically. (By "solve", we mean determine an explicit equation for ζ that is a function of space and time). This is because the equation is non-linear: the vorticity is a function of u and v, but so is the advection operator inside the material derivative. Thus, our strategy will be to simplify things further by linearizing 9.1 about a basic (~ background) state. (You have seen linearization before, for example, in the context of the boussinesq and anelastic equations).

First, we decompose the horizontal velocities into a basic state and a perturbation:

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}' \tag{9.2}$$

The requirement for the basic state (as yet to be specified) is that it must be a solution to our equation 9.1. The absolute vorticity can then be written as

$$\zeta_a = f + \zeta_0 + \zeta' \tag{9.3}$$

and thus 9.1 becomes

$$\frac{D}{Dt}(\zeta + f) = \partial_t(\zeta_0 + \zeta') + (u_0 + u')\partial_x(\zeta_0 + \zeta') + (v_0 + v')\partial_y(f + \zeta_0 + \zeta') = 0 \tag{9.4}$$

Step 2: Remove higher-order terms

The whole point of *linearizing* the set of equations around a basic state is that we can then easily neglect the terms that are quadratic or of higher order in perturbations (i.e. throw out terms that are the product of two perturbations or higher). This is only a good next step if the perturbation quantities are small, and we will make this assumption here. That is, e.g.

$$\mathbf{u}_0 >> \mathbf{u}' \quad \text{and} \quad \zeta_0 >> \zeta' \tag{9.5}$$

In this case, the *approximate vorticity equation* becomes:

$$\frac{D}{Dt}(\zeta + f) \approx \partial_t(\zeta_0 + \zeta') + u_0\partial_x(\zeta_0 + \zeta') + u'\partial_x\zeta_0 + v_0\partial_y(f + \zeta_0 + \zeta') + v'\partial_y(f + \zeta_0) = 0 \tag{9.6}$$

or collecting perturbations to the left-hand-side:

$$\partial_t\zeta' + u_0\partial_x\zeta' + u'\partial_x\zeta_0 + v_0\partial_y\zeta' + v'\partial_y(f + \zeta_0) = -\partial_t\zeta_0 - u_0\partial_x\zeta_0 - v_0\partial_y(f + \zeta_0) \tag{9.7}$$

The basic state terms on the right-hand-side may be interpreted as ("external") forcing terms to the perturbation vorticity.

Step 3: Rewrite in terms of the streamfunction ψ

You may think, well hold on, we have a lot of unknowns here and only one equation! This isn't actually correct though! In actuality, for a given basic state the above equation involves only one unknown! That one unknown is the perturbation streamfunction. Recall that the streamfunction is related to the horizontal velocities and the vorticity in the following way:

$$\psi = \psi_0 + \psi', \quad \text{and} \quad (u', v') = (-\partial_y\psi', \partial_x\psi'), \quad \text{and} \quad (u_0, v_0) = (-\partial_y\psi_0, \partial_x\psi_0) \tag{9.8}$$

and

$$\zeta' = \partial_x v' - \partial_y u' = \partial_{xx}\psi' + \partial_{yy}\psi' = \nabla_H^2\psi', \quad \text{similarly} \quad \zeta_0 = \nabla_H^2\psi_0 \tag{9.9}$$

Step 4: Basic state simplifications

If we now assume that the basic state flow is not only bigger than the perturbations, but also independent of time and only has a component in the x-direction (i.e. only a u_0 component, with $v_0 = 0$) that is zonally symmetric (i.e. varies only with y), or in math:

$$u_0 = u_0(y), \qquad v_0 = 0 \quad \Rightarrow \quad \zeta_0 = -\partial_y u_0, \quad \text{and} \quad \partial_t \zeta_0 = 0, \qquad \partial_x \zeta_0 = 0 \tag{9.10}$$

In this case, the absolute vorticity of the basic state is also only a function of y:

$$\zeta_{0,a} = f - \partial_y u_0 \tag{9.11}$$

Taking these simplifications and plugging them into 9.7 leads to

$$\boxed{\partial_t \zeta' + u_0 \partial_x \zeta' + \partial_y \zeta_{0,a} v' = 0} \tag{9.12}$$

or in terms of the perturbation streamfunction:

$$\boxed{(\partial_t + u_0 \partial_x)\nabla_H^2 \psi' + \partial_y \zeta_{0,a} \partial_x \psi' = 0} \tag{9.13}$$

9.2 Wave equation and solution

General wave solutions for this equation may be sought at this point, using a separation of variables. However, straightforward solutions may be obtained by making the further simplifying assumption that *all coefficients are constant.* By coefficients, we mean the basic-state terms in front of the perturbation quantities. It is important to note that we make this assumption mainly for mathematical convenience, and one need not make this assumption to solve the above equations.

- Assuming $u_0 = U =$ constant results in $\zeta_{0,a} = f$.
- Assuming $\partial_y \zeta_{0,a} =$ constant is the β−plane approximation (i.e. $\partial_y \zeta_{0,a} = \partial_y f =$ constant $= \beta$).

The resulting wave equation with constant coefficients then reads:

$$\boxed{\partial_t \zeta' + U \partial_x \zeta' + \beta v' = 0} \tag{9.14}$$

and for the streamfunction

$$\boxed{(\partial_t + U \partial_x)\nabla_H^2 \psi' + \beta \partial_x \psi' = 0} \tag{9.15}$$

Quick refresher for wave forms (https://en.wikipedia.org/wiki/Sine_wave).

Linearized partial differential equations (PDEs) with constant coefficients, as our equations above, lead to *plane wave solutions*. In our 2-dimensional case:

$$\psi'(\mathbf{x}, t) = \Re\{\Psi \exp(i(\mathbf{K} \cdot \mathbf{x} - \omega t))\} = \Psi_R \cos(kx + ly - \omega t) - \Psi_I \sin(kx + ly - \omega t) \tag{9.16}$$

which can be shown by recalling that $e^{i\theta} = (\cos\theta + i\sin\theta)$ and writing

$$\psi'(\mathbf{x}, t) = \Re\{\Psi \exp(i(\mathbf{K} \cdot \mathbf{x} - \omega t))\} = \Re\{(\Psi_R + i\Psi_I)(\cos(kx + ly - \omega t) + i \sin(kx + ly - \omega t))\} \tag{9.17}$$

with

- $\Psi = \Psi_R + i\Psi_I$ (the constant, and complex wave amplitude)
- $\mathbf{x} = (x, y)$
- ω (the constant *wave frequency*, in units of radians/time)
- $\mathbf{K} \equiv (k, l)$ (the constant wave number vector, in units of radians/length)
- the wave numbers are given by $k \equiv 2\pi/\lambda_x$, $l \equiv 2\pi/\lambda_y$ with (λ_x, λ_y) the wavelengths
- the phase $\phi = \phi(\mathbf{x}, t) = \mathbf{K} \cdot \mathbf{x} - \omega t = kx + ly - \omega t$, seen also by writing the perturbation streamfunction as $\psi' = \Re\{\Psi \exp(i\phi)\}$

In our 2-D case, lines of constant phase can be written as

$$\phi = \text{constant} = kx + ly - \omega t \quad \text{or} \quad y = -\frac{k}{l}x + \frac{\omega}{l}t + \text{constant} \tag{9.18}$$

The second form of this equation is an equation for a line, that is, it is a linear relationship. In 3-D, you would get a linear planar relationship for the surfaces of constant phase. This is where the term "plane waves" get their name.

Plane waves: https://en.wikipedia.org/wiki/Plane_wave.

If 9.16 is truly a solution to 9.15, then we can plug 9.16 into 9.15 and show that the equality holds. Rather than having to carry-around the $\Re$ part, we can simply work with the complex solution ($\psi' \sim \exp(i\phi)$) and then take the $\Re$ part as necessary. Note that one can only do this for linear problems! For nonlinear problems you run into the problem that, for example, the real part of the product of two complex numbers is generally not the product of the real parts.

Continuing forward, it is helpful to recall the following:

$$\partial_x \psi' = ik\psi', \qquad \nabla_H^2 \psi' = -(k^2 + l^2)\psi', \quad \Rightarrow \tag{9.19}$$

$$\partial_x(\nabla_H^2 \psi') = -ik(k^2 + l^2)\psi', \quad \text{and} \quad \partial_t \nabla_H^2 \psi' = i\omega(k^2 + l^2)\psi' \tag{9.20}$$

Inserting the solution into 9.15 therefore gives:

$$i\omega(k^2 + l^2)\psi' + -Uik(k^2 + l^2)\psi' + \beta ik\psi' = 0 \tag{9.21}$$

$$[\omega(k^2 + l^2) + -Uk(k^2 + l^2) + \beta k]\psi' = 0 \tag{9.22}$$

Since $\psi' = 0$ is a trivial solution, the expression in square brackets must then be zero. In this case, we can solve for ω and obtain:

$$\boxed{\omega = Uk - \frac{\beta k}{k^2 + l^2} \quad \text{or} \quad \hat{\omega} = \omega - Uk = -\frac{\beta k}{k^2 + l^2}} \tag{9.23}$$

Thus is the famous *Rossby wave dispersion relation*, where $\hat{\omega}$ is the intrinsic (non-Doppler shifted) frequency. As with all dispersion relationships, this equation relates properties of the wave to one another, i.e. waves of different wave number (i.e. wave lengths) have different frequencies and propagate at different speeds according to this relation.

9.3 Wave kinematics refresher (Vallis Chapter 5 Appendix)

9.3.1 Phase speed

Recall that our plane wave solution was

$$\psi'(\mathbf{x}, t) = \Re\{\Psi \exp(i(\mathbf{K} \cdot \mathbf{x} - \omega t))\} \tag{9.24}$$

If we consider, for simplicity, the one dimensional case of a wave traveling in the $x-$direction only. Then our solution simplifies to

$$\psi'(\mathbf{x}, t) = \Re\{\Psi \exp(i(kx - \omega t))\} = \Re\{\Psi \exp(ik(x - c_p^x t))\} \tag{9.25}$$

where

$$\boxed{c_p^x = \omega/k} \tag{9.26}$$

From these two equations one can see that the phase of the wave (ϕ) propagates at the speed c_p^x and we call this speed the *phase speed*. Similarly, the phase speed in the y-direction would be $c_p^y = \omega/l$. Generalizing to 2-dimensions, a wave traveling in the **K** direction would have phase speed . That is, in general the phase speed is calculated as

$$c_p = \frac{\omega}{|\mathbf{K}|} = \frac{\omega}{(k^2 + l^2)^{1/2}} \tag{9.27}$$

The *phase speed* is the speed at which points of constant phase propagate. For example, consider a monochromatic wave (of one frequency), and the phase speed would tell you the speed at which the crests and troughs propagate.

Wikipedia phase speed: https://en.wikipedia.org/wiki/Phase_velocity

9.3.2 Group velocity

Now consider a wave packet (a wave disturbance of finite extent) made of the superposition of many monochromatic (single-wavelength) waves. An example of a wave packet is outlined in red in the figure below.

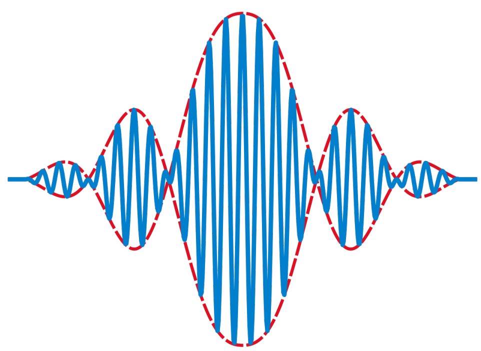

Figure: A wave packet (red) made of a superposition of waves of multiple frequencies (blue). The red "envelope" travels at the group velocity c_g.

The speed at which this packet propagates, termed the *group velocity*, c_g is not the same as the phase velocity c_p. The group velocity instead is defined as

$$\boxed{\vec{c_g} \equiv \nabla_{\mathbf{K}} \omega \quad \text{where} \quad \nabla_{\mathbf{K}} \equiv (\partial_k, \partial_l, \partial_m)} \tag{9.28}$$

For one-dimensional waves, this reduces to

$$c_g^x = \frac{\partial \omega}{\partial k} \tag{9.29}$$

Wikipedia group velocity: https://en.wikipedia.org/wiki/Group_velocity

9.4 Properties of Rossby waves

Back to Rossby waves, which as a reminder, have the dispersion relation

$$\omega = Uk - \frac{\beta k}{k^2 + l^2} \tag{9.30}$$

The phase speed of Rossby waves in the x-direction is then given by

$$\boxed{c_p^x = \frac{\omega}{k} = U - \frac{\beta}{k^2 + l^2}} \tag{9.31}$$

or equivalently

$$c_p^x = \frac{\omega}{k} = U - \frac{\beta}{K^2} \quad \text{where} \quad K^2 = k^2 + l^2 \tag{9.32}$$

The intrinsic phase speed, or the phase speed relative to the background flow is

$$c_p^x - U = -\frac{\beta}{k^2 + l^2} = -\frac{\beta}{K^2} \tag{9.33}$$

that is, the intrinsic phase speed of Rossby waves is always negative (westward)! These waves always propagate westward relative to the mean flow. Thus, *bigger waves travel faster to the west!*

The Rossby wave group velocity in the direction of the basic flow is

$$c_g^x = \frac{\partial \omega}{\partial k} = U - \beta \frac{K^2 - 2k^2}{K^4} = U + \beta \frac{k^2 - l^2}{(k^2 + l^2)^2} \tag{9.34}$$

For $|k| > |l|$, $c_g^x > 0$ (eastward), or in the opposite direction of the phase speed (see Group Velocity Wikipedia animation to see an example of this).

The group velocity in the y direction is given by

$$c_g^y = \frac{\partial \omega}{\partial l} = \frac{2\beta k l}{K^4} \tag{9.35}$$

Scales of stationary waves

Stationary waves are waves who's phase lines are stationary relative to the ground. That is, $c_p^x = 0$ and so $k_S^2 = \beta/U$. Using typical midlatitude values for $\beta \sim 10^{-11}$ 1/(ms) and $U \sim 10$ m/s ...

What is the typical stationary wavelength in midlatitudes for $l^2 << k^2$?

$k \equiv 2\pi/\lambda_x$ and assuming $l = 0$, $k^2 = \beta/U$ and therefore $\lambda_x = 2\pi/\sqrt{(\beta/U)} \approx 6000$ km

What is the intrinsic period of these waves (inverse of frequency)?

$1/\hat{\omega} = -k/\beta = -10^{-6}/10^{-11} = -10^5$ seconds per radian, and converting to days per wave cycle $-10^5 \times 2\pi/(60 \times 60 \times 24) \approx 7$ days.

What is the group velocity of these waves?

$c_g^x = U + \beta/k^2 = U + U = 2U$ - that is, the wave group (and therefore energy) propagates eastward at twice the basic state flow! This is related to "downstream development".

9.5 Rossby wave propagation mechanism

Vallis, Ch. 5.7:

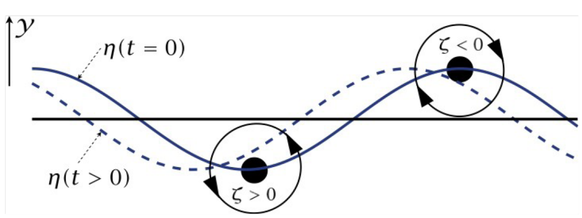

Fig. 5.4 The mechanism of a two-dimensional (x-y) Rossby wave. An initial disturbance displaces a material line at constant latitude (the straight horizontal line) to the solid line marked $\eta(t = 0)$. Conservation of potential vorticity, $\beta y + \zeta$, leads to the production of relative vorticity, as shown for two parcels. The associated velocity field (arrows on the circles) then advects the fluid parcels, and the material line evolves into the dashed line. The phase of the wave has propagated westwards.

Hoskins et al. (1985): *Hoskins, McIntyre, Robinson, On the use and significance of isentropic potential vorticity maps, QJRMS, 1985.*

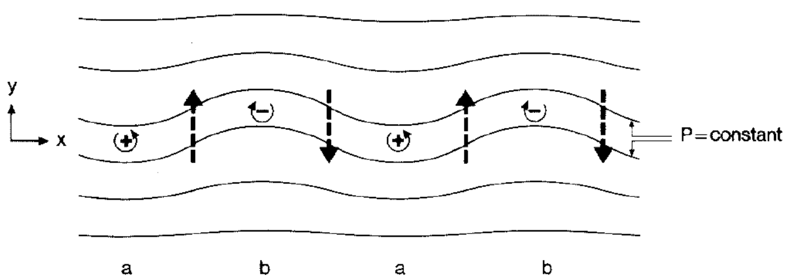

Figure 17. IPV map for a simple, x-periodic Rossby wave. The + and − signs respectively indicate the centres of the cyclonic and anticyclonic IPV anomalies due to southward and northward air-parcel displacements across a basic northward IPV gradient. The heavy, dashed arrows indicate the sense and relative phase of the induced velocity field (see text) which causes the westward propagation of the phase of the pattern.

Figure: Two ways of looking at Rossby wave propagation and why the intrinsic phase speed is always westward.

The intrinsic westward progression of the phase lines of Rossby waves result from the wave perturbations in vorticity acting on the wave itself. In a way, the wave is advecting itself. Note how that $\beta-$ effect is crucial for the Rossby wave mechanism. That is, differential rotation (a changing f) provides the restoring mechanism for Rossby waves.

9.6 Linear waves are nonlinear solution

What happens if we insert the wave solution ψ' into the full non-linear equation?

$$\begin{aligned}
\frac{D(\zeta+f)}{Dt} &= \left[\partial_t + (U+u')\partial_x + v'\partial_y\right]\zeta' + \beta v' \\
&= \left[\partial_t + (U - \partial_y\psi')\partial_x + \partial_x\psi'\partial_y\right]\nabla^2\psi' + \beta\partial_x\psi' \\
&= \left[-\partial_y\psi'\partial_x + \partial_x\psi'\partial_y\right]\nabla^2\psi' \\
&= \partial_x\psi'\left[-l/k\partial_x + \partial_y\right]\nabla^2\psi' \\
&= \partial_x\psi'\nabla^2\left[-l/k\partial_x\psi' + \partial_y\psi'\right] = \partial_x\psi'\nabla^2\left[-l/k\partial_x\psi' + l/k\partial_x\psi'\right] = 0\,.
\end{aligned}$$

The third step above uses the fact that we already know the linear part of the equation is zero, the fourth and fifth steps use $\partial_y\psi' = l/k\partial_x\psi'$. We therefore find that these Rossby waves have the special property (not shared by other waves) that they are also a solution of the full non-linear equations (even though the waves themselves are linear)!

9.7 Insight into the existence of jet-streams (Vallis 12.1)

Consider a Rossby wave source located in the x-y plane. Theory tells us that the energy and wave packet will propagate away from the source (group velocity away from the source), and theory tells us that the wave packet will propagate along great circle routes. For the meridional propagation then, we expect the group velocity to be positive northward of the source and negative southward of the source. That is,

$$c_g^y \equiv \frac{\partial\omega}{\partial l} = \frac{2\beta k l}{(k^2+l^2)^2} \tag{9.36}$$

- northward of the source we require $c_g^y > 0$ and therefore $kl > 0$
- southward of the source we require $c_g^y < 0$ and therefore $kl < 0$

Now consider the meridional flux of zonal momentum (often simply called *the momentum flux*) for barotropic Rossby waves:

$$\overline{u'v'} = -\overline{\partial_x\psi'\partial_y\psi'} \propto -kl \propto -c_{g,y} \tag{9.37}$$

where overbars denote zonal averages (see derivation at the end of this section).

Thus, the meridional flux of zonal momentum is in the opposite direction of the group velocity in the meridional direction! This result is quite profound. It tells us that as the waves propagate *away* from the source region (the midlatitudes), eastward zonal momentum will be fluxed back into the midlatitudes (see figure below)! This is in essence why we have westerlies/jet-streams in the midlatitudes. Moreover, this is why the midlatitude jet-stream is referred to as the *midlatitude, eddy-driven jet stream*.

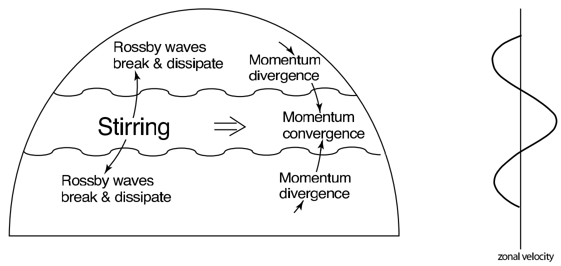

Fig. 12.4 Generation of zonal flow on a β-plane or on a rotating sphere. Stirring in midlatitudes (by baroclinic eddies) generates Rossby waves that propagate away from the disturbance. Momentum converges in the region of stirring, producing eastward flow there and weaker westward flow on its flanks.

Figure: Illustration from Vallis (Chapter 12) of Rossby waves propagating away from a midlatitude source and fluxing momentum back into the midlatitudes to form a zonal jet-stream.

Derivation of 9.37

We recall that the solution for the perturbation streamfunction can be written as:

$$\psi' = \Re\{(\Psi_R + i\Psi_I)(\cos\phi + i\sin\phi)\} \tag{9.38}$$

where $\phi = (kx + ly - \omega t)$ to make notation easier and Ψ is a complex constant. Then

$$u' = -\frac{\partial\psi'}{\partial y} = -\Re\{(\Psi_R + i\Psi_I)il(\cos\phi + i\sin\phi)\} = l(\Psi_R\sin\phi + \Psi_I\cos\phi) \tag{9.39}$$

$$v' = \frac{\partial\psi'}{\partial x} = \Re\{(\Psi_R + i\Psi_I)ik(\cos\phi + i\sin\phi)\} = -k(\Psi_R\sin\phi + \Psi_I\cos\phi) \tag{9.40}$$

Plugging this in we get

$$\overline{u'v'} = -kl\overline{\Psi_R^2\sin^2\phi + \Psi_I^2\cos^2\phi + 2\Psi_R\Psi_I\cos\phi\sin\phi} \tag{9.41}$$

$$= -kl\left(\Psi_R^2\overline{\sin^2\phi} + \Psi_I^2\overline{\cos^2\phi} + 2\Psi_R\Psi_I\overline{\cos\phi\sin\phi}\right) \tag{9.42}$$

Recalling that the overbar denotes a zonal mean, and assuming periodic boundary conditions around a latitude circle,

$$\overline{\sin^2\phi} = \overline{\cos^2\phi} = \frac{1}{2} \quad \text{and} \quad \overline{\sin\phi\cos\phi} = 0 \tag{9.43}$$

leads to

$$\overline{u'v'} = -\frac{1}{2}\Psi^2 kl \propto -kl \tag{9.44}$$